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Research articles

Didactical design of the proportionality concept based on anthropological theory of the didactic

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Abstrak Penelitian ini bertujuan untuk mengembangkan desain didaktik berbasis Teori Antropologi Didaktik (ATD) dengan fokus pada praksiologi matematika untuk membantu siswa agar lebih memahami konsep proporsionalitas. Metode yang digunakan adalah Didactical Design Research (DDR), yang terdiri dari tiga tahap utama: analisis retrospektif, analisis metadidaktik, dan analisis situasi didaktik prospektif. Data dikumpulkan melalui observasi di kelas, wawancara dengan guru dan siswa, serta analisis tugas diagnostik yang dikerjakan oleh 30 siswa kelas VII di sebuah SMP di Jawa Barat. Menggunakan kerangka Brousseau hasil penelitian ini menunjukkan bahwa hambatan belajar siswa terbagi menjadi tiga jenis: (1) hambatan ontogenik, yaitu terkait kesiapan dan pengetahuan awal siswa; (2) hambatan epistemologis, seperti penggunaan penalaran aditif yang salah dalam konteks yang seharusnya menggunakan penalaran multiplikatif, serta kesalahan dalam memahami laju satuan; dan (3) hambatan didaktik, yang muncul karena struktur tugas yang asing, informasi yang tidak jelas, serta kurangnya representasi yang memadai dalam pembelajaran. Berdasarkan temuan tersebut, dirancang lima tugas pembelajaran melalui pendekatan praksiologi. Studi ini memberikan alternatif strategi bagi guru untuk mengatasi kesulitan belajar proporsionalitas dan mendukung terciptanya pembelajaran matematika yang lebih bermakna dan relevan dengan konteks siswa. Selain itu, penelitian ini juga membuka peluang untuk penelitian lebih lanjut terkait penerapan desain ini dan dampaknya terhadap cara berpikir matematika siswa.

Kata kunci *Teori antropologi didaktik, Praksiologi matematika, Proporsionalitas*

Abstract This study aims to develop a didactic design grounded in the Anthropological Theory of the Didactic (ATD), with a focus on mathematical praxeology, to enhance students' understanding of proportionality. The study uses the Didactical Design Research (DDR) methodology, which consists of three primary stages: retrospective analysis, metadidactical analysis, and prospective didactical situation analysis. Classroom observations, teacher and student interviews, and the examination of diagnostic tasks completed by thirty seventh-grade students at a junior high school in West Java, Indonesia, were used to gather data. The findings, which are interpreted using the Brousseau's framework, classify learning obstacles into three categories: (1) Ontogenic obstacles, which are associated with students' developmental readiness and prior knowledge; (2) Epistemological obstacles, which include misusing additive reasoning in multiplicative contexts and misinterpreting unit rates; and (3) Didactic obstacles, which are caused by unfamiliar task structures, implicit information, and insufficient instructional representations. The study has developed five learning tasks based on the praxeological framework, which included task, technique, technology, and theory. Despite not yet being used in classrooms, the didactic design's development is supported by both theoretical and empirical evidence. This study offers a different strategy for teachers dealing with proportionality learning obstacles and helps create more meaningful and contextually relevant math instruction. It also creates opportunities for further study to examine how the suggested design is implemented and how it affects students' mathematical thinking.

Keywords *Anthropological theory of the didactic, Mathematical praxeology, Proportionality*

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Introduction

Since mathematics is an abstract and ever-evolving field, it necessitates the application of pedagogically sound and theoretically supported teaching strategies. Among these, didactic methods are crucial for both imparting mathematical knowledge and creating engaging learning opportunities that could help fill the gap between students' conceptual understanding and formal knowledge. Converting complicated scientific knowledge into formats that can be effectively taught and understood in classroom settings is one ongoing challenge in this area (Chevallard, 1991). This difficulty emphasizes the need for didactic techniques that are grounded in solid theoretical frameworks and have the ability to mediate between students' cognitive structures and institutional knowledge.

The Anthropological Theory of the Didactic (ATD), created by Chevallard (1989), is a potential framework that tackles this problem. ATD offers a thorough viewpoint on how knowledge is arranged and disseminated in educational environments. The idea of didactic transposition, or the process by which scholarly knowledge (savoir savant) is converted into knowledge to be taught (savoir enseigné) and subsequently into knowledge that students actually learn (savoir appris), is fundamental to ATD. ATD incorporates this idea as one of its primary analytical tools rather than just endorsing didactic transposition. Praxology, a systematic model for examining human behavior in educational settings, is another essential element of ATD. Four elements make up a praxeology: the type of task, the techniques used to solve it, the rationale behind these techniques (technology), and the theoretical justification (theory) (Chevallard, 2006).

In order to analyze learning barriers and create didactic interventions, this study uses ATD, specifically its praxeological model, as a lens. The idea of proportionality, which represents an area where the didactic transposition of mathematical concepts faces major difficulties, is the subject of the discussion. Many higher-level concepts, including ratios, rates, linear relationships, and algebraic reasoning, are based on the fundamental mathematical concept of proportionality (Van de Walle, Karp, & Bay-Williams, 2022). However, its instruction frequently suffers from inadequate didactic design, particularly when moving from concrete-number arithmetic operations to abstract algebraic representations (Radford, 2014). Despite extensive reporting, instructional practices continue to fall short in addressing this gap (Hackenberg & Lee, 2015; Prediger & Zindel, 2017).

For example, proportional reasoning is frequently introduced through isolated numerical problems without enough emphasis on multiplicative structure, visual representation, or context-based reasoning, according to an analysis of textbook tasks and classroom practices in Indonesian primary schools (Charalambous, 2010; Maudy & Ruli, 2024). This leads to persistent learning barriers and fragmented understanding. Examples include misinterpreting unit rates, particularly when dealing with decimal values or non-integer ratios, failing to distinguish between proportional and non-proportional situations, and abusing additive strategies in multiplicative contexts (Hohensee, et al., 2016; Wijaya et al., 2021). When pupils are exposed to algebraic symbolism too soon before they have a firm understanding of proportional relationships, these difficulties are made worse (Bofferding & Wessman-Enzinger, 2018).

Our initial study's empirical results shed additional light on these problems. A diagnostic tool consisting of eight contextualized tasks given to 30 seventh-grade students allowed us to identify five main types of learning barriers: (1) incorrect additive strategies; (2) incorrect multiplicative factor interpretations; (3) inappropriate unit rate use; (4) improper proportional reasoning in non-proportional contexts; and (5) a general lack of quantitative reasoning (Maudy & Ruli, 2024). A basic misunderstanding of the multiplicative relationship between quantities was demonstrated, for instance, when some students computed results by directly adding time differences to energy values in a task that involved energy consumption over time (González-Forte et al., 2020).

In light of these results, a didactic design that not only overcomes students' learning challenges but also reimagines proportionality instruction in a way that is relevant, contextual, and theoretically grounded is desperately needed. By recognizing pertinent problem types, anticipating student strategies, and coordinating these with logical mathematical theories, the praxeological approach provides a promising solution by allowing the explicit design of instructional tasks (Bosch & Gascón, 2006). Under this framework, learning becomes more than just following instructions; it also involves comprehending the reasoning behind them, which is essential for developing profound conceptual knowledge (Prediger et al., 2015).

According to recent studies, praxeology-based instructional designs are effective at removing learning obstacles, especially when it comes to early algebra and proportionality (Maudy, 2023). Even though these studies have established a solid foundation, they frequently don't provide thorough explanations of how learning barriers are methodically found or how contextual task designs are directly linked to students' cognitive requirements (Clarke, 2021). Furthermore, previous research has mostly focused on conventional fraction-based proportional problems, ignoring other important task types like non-routine comparisons, missing-value scenarios with irrational ratios, and qualitative reasoning tasks involving rate of change (Van den Heuvel-Panhuizen, 2003; Lamon, 2007, Jitendra et al., 2020).

This study suggests a didactic design based on ATD and organized through a praxeological lens to fill in these gaps. It is supported by empirical data showing that students actually struggle with proportional reasoning. The study's goals are to: (1) determine and examine the learning challenges that students face when comprehending proportionality; (2) use praxeological analysis to create a task sequence that addresses these challenges; and (3) assess how well contextualized task designs can support students' conceptual understanding.

A validated didactic design model for proportional reasoning that can be modified for use in primary mathematics education and that methodically tackles learning challenges and facilitates students' progression from arithmetic to early algebraic thinking is the anticipated result of this study. In addition to advancing didactic design research, this contribution is expected to offer useful advice to educators and curriculum designers who want to enhance proportionality instruction in actual classroom environments.

Methods

The three main phases of this study's qualitative methodology, which is based on Didactical Design Research (DDR), are (1) prospective didactical situation analysis, (2) metadidactical analysis, and (3) retrospective analysis (Artigue, 2009). These phases offer a methodical framework for creating, putting into practice, and improving didactic interventions meant to help students overcome their learning challenges related to proportionality. This study incorporates Chevallard's (1989, 2006) mathematical praxeology principles into the DDR framework to guarantee conceptual coherence.

Thirty seventh-grade students and two math teachers participated in the study, which was carried out at a lower secondary school in Indonesia. A public junior high school in the West Java Province served as the site of data collection. The selection of study participants was based on their willingness to participate and their accessibility.

The concept of proportionality was the subject of an epistemological analysis in the first phase, known as prospective analysis. This required analyzing academic knowledge from advanced mathematics textbooks and determining the praxeological structures that were pertinent to the subject. The current article focuses specifically on the praxeological development and the design of didactic sequences to address proportionality-related learning obstacles, even though it draws from previous analyses of curriculum and teaching practices.

The researchers created an instrument for identifying learning obstacles based on the prospective analysis. It consists of eight problem scenarios. These tasks were developed according to key indicators derived from the prevalent misconceptions and cognitive challenges documented in the literature (Maudy & Ruli, 2024). Students' comprehension in areas like incorrect use of additive strategies, improper application of multiplicative comparisons, misuse of unit rates, and misidentification of proportional versus non-proportional contexts were evaluated by the problems.

In order to gather information about how students interacted with the learning tasks, classroom observations were used in the second phase. Semi-structured interviews with teachers and students were conducted after these observations in order to confirm the learning barriers and triangulate the issues noted. To bolster the analysis, sample student work and classroom artifacts were also gathered. With codes generated both inductively from student responses and interactions and deductively from the praxeological model (tasks, techniques, technology, and theory), the data were analyzed thematically.

Retrospective analysis was used in the third phase to improve the original didactic design in light of the empirical data obtained in the second phase. In order to update the teaching resources and improve the correspondence between task design and students' conceptual development, this phase integrated metadidactical reflections between researchers and educators.

Based on the results of these analyses, the study designs a didactic framework that incorporates the four main components suggested by the principles of mathematical praxeology (Chevallard, 1989, 2006):

1. Tasks: Design relevant and contextual tasks aimed at fostering students' understanding of the concept of proportionality. These tasks are designed to engage students in mathematical thinking activities, considering their cognitive development levels. The tasks should also be sufficiently challenging to encourage critical thinking and help students overcome the learning obstacles they encounter.
2. Techniques: Develop techniques that students can use to solve these tasks. These techniques are designed to avoid common errors that constitute learning obstacles, such as the improper application of additive methods or confusion in understanding ratios and proportions. The design should accommodate various problem-solving techniques that students can use to address proportionality problems.
3. Technology: Provide arguments or explanations that connect the techniques used to solve the tasks with the foundational principles underlying them, allowing students to develop a deeper understanding of the concept of proportionality.

4. Theory: General mathematical concepts used to justify the various technologies. This provides the necessary theoretical foundation to explain the technologies employed.

This comprehensive approach ensures that the instructional design is grounded in both theory and empirical data, allowing for iterative improvement and contextual relevance.

Findings and Discussion

This section presents the research findings, which consist of empirical data accompanied by data analysis that leads to the discovery of key insights. Scholarly knowledge is considered *a priori* knowledge; therefore, it must undergo a process of transposition into curriculum content—*knowledge to be taught*. The primary school mathematics curriculum is transposed into instructional content within the framework of teaching and learning activities, referred to as *taught knowledge*. Furthermore, an analysis was conducted on students' learning obstacles related to the concept of proportionality at the secondary school level.

To address these challenges, the researcher developed instructional content in the form of a didactical design, which comprises two main components: praxis (including the types of tasks and techniques) and logos (comprising the technology and the underlying theoretical foundation). The didactical design introduces a sequence of problem situations (*types of tasks*) accompanied by appropriate techniques used to solve them. It also includes the technology, which refers to the justification of the techniques employed, and the theory, representing the overarching mathematical concepts used to validate the technology.

Analysis of learning obstacles for the concept of proportionality

This section presents an analysis of students' learning obstacles related to the concept of proportionality at the secondary school level, based on specific problems provided during the study. The researcher also outlines these problems through data obtained from observations, interviews, and analysis of instructional documents.

Problem 1

The task type in Problem 1 can be classified as a "missing value" problem. It involves a familiar context—nutritional labels on food products—where the unit rate is not explicitly stated. This may lead students to be misled by the absence of a whole-number scale factor between the given ratio pairs.

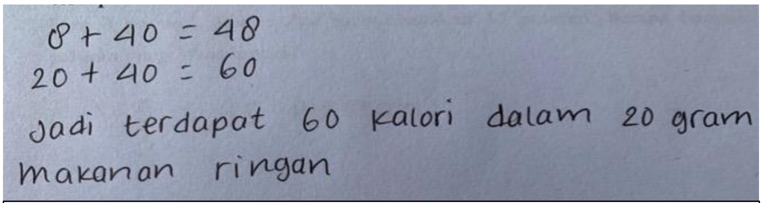
Task

There are 48 calories in 8 grams of a snack. In 16 grams, there are 96 calories. How many calories are there in 20 grams of the snack?

Learning obstacle

The researcher identified specific learning obstacles experienced by students in relation to Problem 1.

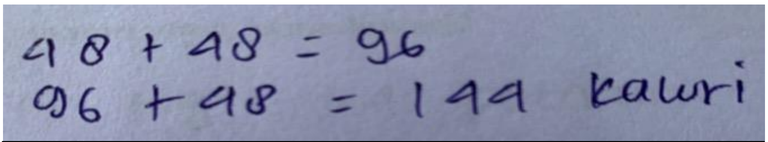
The first identified learning obstacle involves the incorrect use of an additive strategy based on the initial gram-calorie pair. The student attempted to solve the problem by applying the method: (number of grams) + 40 calories. As shown in Figure 1, the student erroneously used an additive reasoning in the first statement, interpreting it as 8 + 40 = 48, which subsequently led to an incorrect response to the question—namely, 20 + 40 = 60. The student overlooked the second statement in the problem, which indicates that 16 grams correspond to 96 calories, and failed to verify their strategy accordingly, as 16 + 40 ≠ 96. This indicates that the student did not utilize all the provided information, ultimately arriving at an incorrect conclusion.



8 + 40 = 48
20 + 40 = 60
So there are 60 calories in 20 grams of snacks.

Figure 1. Learning obstacle 1 in problem 1

The second learning obstacle involves a flawed extension of the scale factor approach. The student employed additive reasoning by observing that the calorie count increases by 48 when the quantity of the snack doubles, from 8 grams to 16 grams. From this observation, the student incorrectly inferred that 96 + 48 (i.e., 48 + 48 + 48 or 48 × 3) corresponds to a threefold increase in grams—namely, 8 grams × 3 = 24 grams of the snack—which does not match the 20 grams asked in the question. However, as shown in Figure 2, the student proceeded with the calculation: 48 + 48 = 96, followed by 96 + 48 = 144, leading to an incorrect solution.



48 + 48 = 96
96 + 48 = 144 calories

Figure 2. Learning obstacle 2 in problem 1

The third learning obstacle is a misapplication of the scale factor approach. This approach is one of the intuitive methods students often use when engaging in proportional reasoning. It involves the use of scalar multiplication within a measurable space—doubling, tripling, and so on are all scalar operations. However, a constant scalar factor is not always required. As seen in Figure 3, the student performed the calculations: 48 × 2 = 96, followed by 96 × 2 = 192. This indicates that the student multiplied the number of calories in proportion to the increase in grams, using a doubling factor based on 16 ÷ 8 = 2. However, this scalar is not consistent across all known quantities of grams. Instead, the student should have applied a different constant factor—either 20 ÷ 8 using the first data point, or 20 ÷ 16 using the second. It is likely that the student was misled by the absence of a whole-number scale factor between the given ratio pairs.

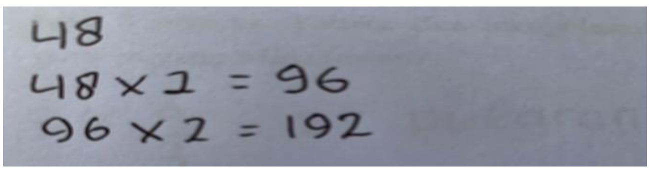


Figure 3. Learning obstacle 3 in problem 1

Technique

A correct technique is demonstrated in the student's response shown in Figure 4, which employs the unit rate approach. The unit rate is determined through a straightforward division across the measurement space and is then used as a multiplier to calculate the total number of calories. In this case, the unit rate is identified as 6 calories per gram. By multiplying this rate by 20 grams, the student correctly calculated the total to be 120 calories.

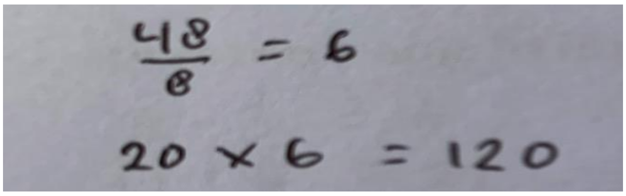


Figure 4. Student's response to problem 1

Problem 2

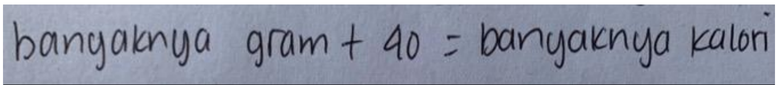
The second task type involves generalization into a rule of the form $y = mx$, which can be applied to determine any rate pair within the given proportional relationship associated with the context. The context used is a familiar one involving unit rates. However, the task is presented in a non-routine format, and students may be unfamiliar with the notation potentially involved in the solution.

Task

There are 48 calories in 8 grams of a snack. In 16 grams, there are 96 calories. What rule can be used to determine the number of calories in each gram of the snack?

Learning obstacle

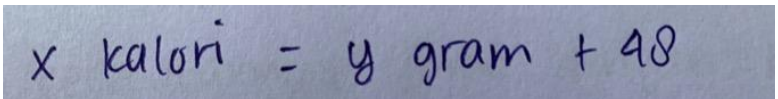
The researcher identified specific learning obstacles encountered by students in relation to Problem 2. The first learning obstacle involves an incorrect generalization of the additive approach to the problem, represented by the formulation: (⊡ grams) + 40 = (⊡ calories). As shown in Figure 5.



number of grams + 40 = number of calories

Figure 5. Learning obstacle 1 in problem 2

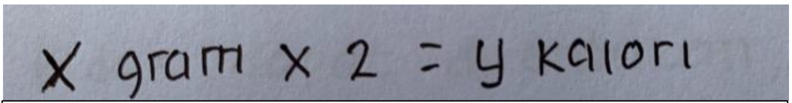
The second learning obstacle is a flawed generalization of the additive approach to the problem, wherein the student interprets the relationship as 48 calories per 8 grams and formulates it as: (⊡ grams) + 48 = (⊡ calories). As shown in Figure 6.



x calories = y gram + 48

Figure 6. Learning obstacle 2 in problem 2

The third learning obstacle is an incorrect generalization of the integer scale factor approach. In this case, both the number of grams and the number of calories are doubled simultaneously—8 grams × 2 = 16 grams, 48 calories × 2 = 96 calories—leading the student to generalize the relationship as: (⊡ grams) × 2 = (⊡ calories), as shown in Figure 7.

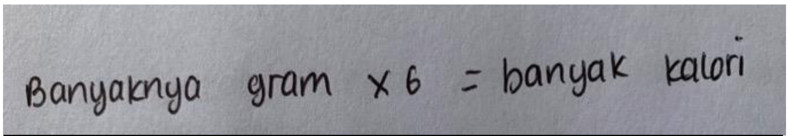


x gram × 2 = y calories

Figure 7. Learning obstacle 3 in problem 2

Technique

The correct technique involves using the invariant unit rate of 6 calories per 1 gram to express the relationship in the form $y = mx$, such as $y = 6x$. Therefore, 48 calories = 6 × 8 grams and 96 calories = 6 × 12 grams. This leads to the generalization, as shown in the student's response in Figure 8: (⊡ grams) × 6 = (⊡ calories).



number of grams × 6 = number of calories

Figure 8. Student's response to problem 2

Problem 3

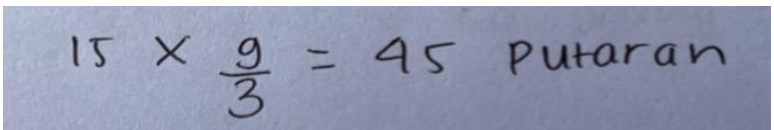
The third task type is a non-proportional problem. The context involves a familiar scenario of running laps.

Task

Mina and Zea run at the same speed around a track. Mina starts running first. When Mina has completed 9 laps, Zea has only completed 3 laps. When Zea finishes 15 laps, how many laps will Mina have completed?

Learning obstacle

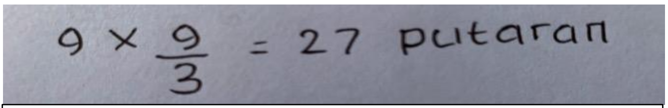
The researcher identified specific learning obstacles encountered by students in relation to Problem 3. The first learning obstacle involves the application of proportional reasoning to a situation that is, in fact, non-proportional. As shown in the student's response in Figure 9, the student applied a proportional strategy: 15 × (9⁄3) = 45, which leads to an incorrect conclusion.



$15 \times \frac{9}{3} = 45$ laps

Figure 9. Learning obstacle 1 in problem 3

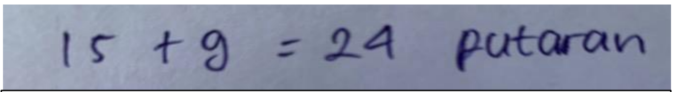
The second learning obstacle also reflects the incorrect application of proportional reasoning to a non-proportional situation. As illustrated in the student's response in Figure 10, the student used the following calculation: $9 \times (9/3) = 27$, which again results in a mathematically invalid conclusion given the context.



$9 \times \frac{9}{3} = 27$ laps

Figure 10. Learning obstacle 2 in problem 3

The third learning obstacle involves incorrect additive reasoning. The student assumed that Mina had consistently run 9 laps more than Zea throughout, and therefore concluded that when Zea completed 15 laps, Mina had completed $15 + 9 = 24$ laps, as shown in Figure 11.



15 + 9 = 24 laps

Figure 11. Learning obstacle 3 in problem 3

Technique

The correct technique involves recognizing that Mina ran 6 laps more than Zea. Therefore, when Zea completed 15 laps, Mina had completed $15 + 6 = 21$ laps. As shown in Figure 12.

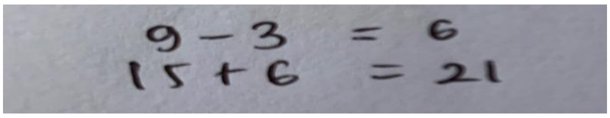


Figure 12. Student's response to problem 3

Problem 4

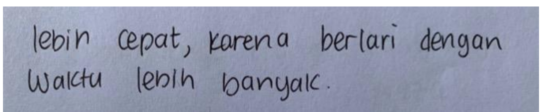
Task type 4 involves qualitative reasoning. The context used is a familiar one related to speed, and the task is presented as a non-routine problem.

Task

Bani runs around a track every day. He runs fewer laps in a longer time today than he did yesterday. Is his running speed today faster, slower, the same as yesterday, or indeterminate?

Learning obstacle

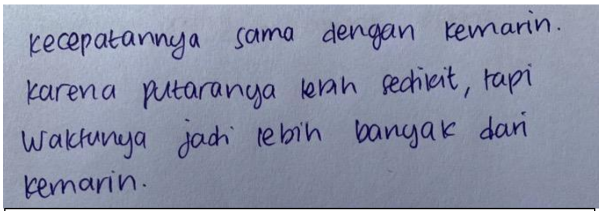
The researcher identified specific learning obstacles encountered by students in relation to Problem 4. The first learning obstacle involves reversed reasoning. The student's explanation for their answer was that being able to run for a longer time translates to running faster, as shown in Figure 13.



Faster, because running with more time.

Figure 13. Learning obstacle 1 in problem 4

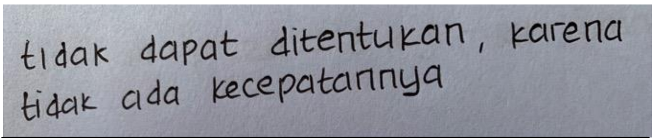
The second learning obstacle involves illogical reasoning. The student did not engage in reasoning based on a pattern or a logical understanding of speed as a quantity. The student assumed that the speed remained the same, reasoning that by reducing the distance and increasing the time, these changes balance each other out, resulting in the same speed, as shown in Figure 14.



The speed is the same as yesterday because there are fewer laps, but the time has become more than yesterday.

Figure 14. Learning obstacle 2 in problem 4

The third learning obstacle involves the student's failure to engage in an analytical thought process. The student responded with "cannot be determined" (Figure 15). The student believed that a specific numerical value must be provided to solve the problem, or that it is an indeterminate case. Speed is a quantity that relates to both distance and time. Exploring qualitative information involves understanding relationships or changes in these quantities without the need for accompanying numerical values. The student was unable to analyze the qualitative information in order to solve the problem.

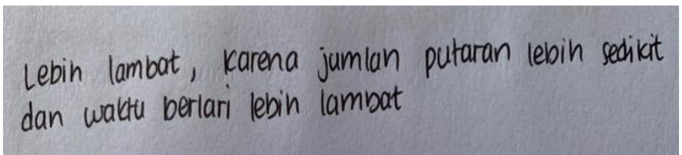


Cannot be determined, since there is no speed.

Figure 15. Learning obstacle 3 in problem 4

Technique

The correct technique involves the student using proportional reasoning. The student correctly concluded that Bani was running slower. By reducing the amount of distance (laps) covered and increasing the running time, Bani is running at a slower speed.



Slower, because the number of laps is less and the running time is slower.

Figure 16. Student's response to problem 4

Problem 5

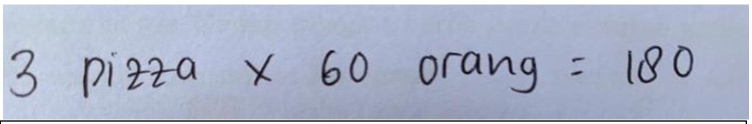
Task type 5 involves a missing value problem. The context used is a familiar one related to pizza ordering, where the numbers provided facilitate the use of the scale factor approach.

Task

Diana is ordering pizza for a birthday party. She estimates that 3 pizzas will be enough for 10 people. How many pizzas should Diana order for 60 people?

Learning obstacle

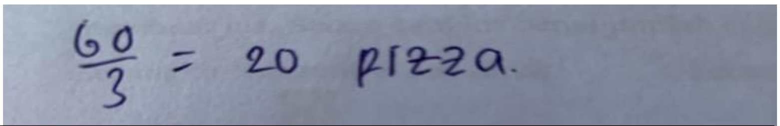
The researcher identified specific learning obstacles encountered by students in relation to Problem 5. The first learning obstacle involves an incorrect interpretation of the unit rate. The student mistakenly interpreted the number of pizzas as a unit rate of pizzas per person, leading to the calculation 180 = 3 pizzas × (60 people), as shown in the student's response in Figure 17.



3 pizzas × 60 people = 180

Figure 17. Learning obstacle 1 in problem 5

The second learning obstacle involves an incorrect scale factor approach, where the quantity of people is divided by the quantity of pizzas, resulting in 20 = 60 ÷ 3. This leads to an incorrect ratio, as it uses two different ratio pairs, leading to a flawed calculation. As shown in Figure 18.



$\frac{60}{3}$ = 20 pizzas

Figure 18. Learning obstacle 2 in problem 5

The third learning obstacle involves the student failing to apply the identified scale factor to solve the problem. As shown in the student's response in Figure 19, 60 ÷ 10 = 6, which is an integer in the scale factor space of change. The scale factor was correctly identified but was not subsequently used to solve the problem.

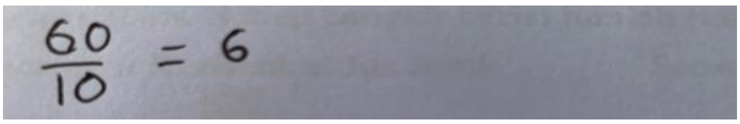
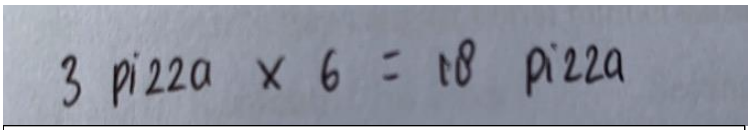


Figure 19. Learning obstacle 3 in problem 5

Technique

The correct technique is demonstrated by the student's response in Figure 20, where 3 pizzas × 6 = 18 pizzas. The use of the scale factor in the "people" dimension is extended into the "pizza" dimension.



3 pizzas × 6 = 18 pizzas

Figure 20. Student's response to problem 5

Problem 6

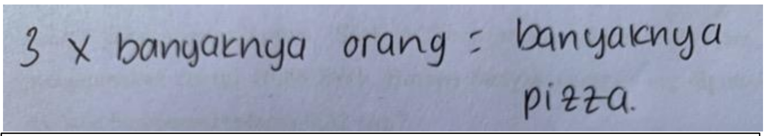
Task type 6 involves a non-routine problem. There may be unfamiliar notation involved. The previous problem guided students to think about using the scale factor approach in a rule. Students need to be able to navigate this complexity.

Task

Diana is ordering pizza for a birthday party. She estimates that 3 pizzas will be enough for 10 people. What rule can be used to determine the number of pizzas Diana needs to order for a given number of people?

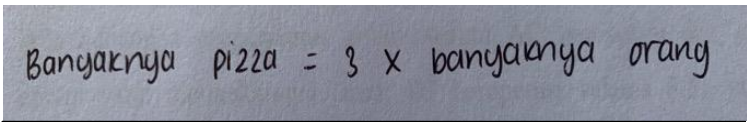
Learning obstacle

The researcher identified specific learning obstacles encountered by students in relation to Problem 6. The first learning obstacle involves the incorrect use of the unit rate. The student incorrectly used the information "3 pizzas per person" as the unit rate between the given dimensions, when it should have been "3 pizzas per 10 people." As shown in Figure 21 and Figure 22.



3 × number of people = number of pizzas

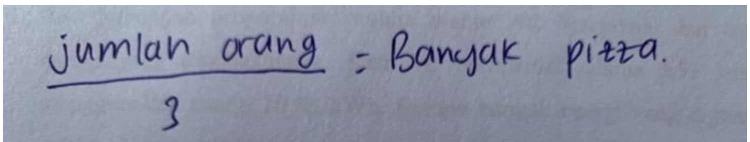
Figure 21. Learning obstacle 1a in problem 6



number of pizza = 3 × number of people

Figure 22. Learning obstacle 1b in problem 6

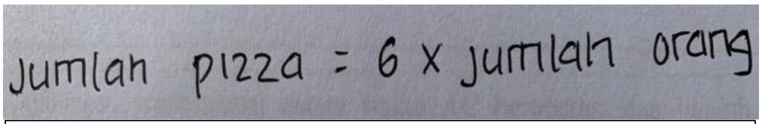
The second learning obstacle involves the use of a different unit rate in an incorrect context. The student used the reciprocal of the unit rate, which should not have been applied in this rule. As shown in Figure 23.



$\frac{\textit{number of people}}{3}$ = number of pizzas

Figure 23. Learning obstacle 2 in problem 6

The third learning obstacle involves the use of a scale factor in an incorrect context. The scale factor from the previous task was incorrectly applied as a replacement for the invariant unit rate. As shown in Figure 24.



number of pizzas = 6 × number of people

Figure 24. Learning obstacle 3 in problem 6

Technique

The technique used is the application of the invariant unit rate, i.e., 3/10, so the number of pizzas is calculated as:
(# Pizza) = (3 pizzas per 10 people) × (# People).

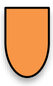
Problem 7

Task type 7 involves a comparison problem. The context used is a familiar one involving the mixing of orange juice concentrate and water. The task presented is non-routine. The ordered pairs with corresponding terms are multiples of each other.

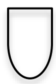
Task

Orange juice is made by mixing orange juice concentrate and cups of water. The recipe is described below in terms of the number of cups of orange juice concentrate and the number of cups of water mixed together to make the juice. Each cup contains the same amount of liquid.

One cup of orange juice concentrate

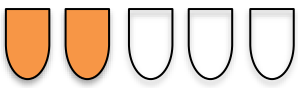


One cup of water

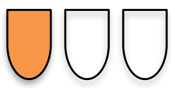


Which recipe has the strongest orange flavour, do the recipes have the same flavour, or can it not be determined?

Recipe A

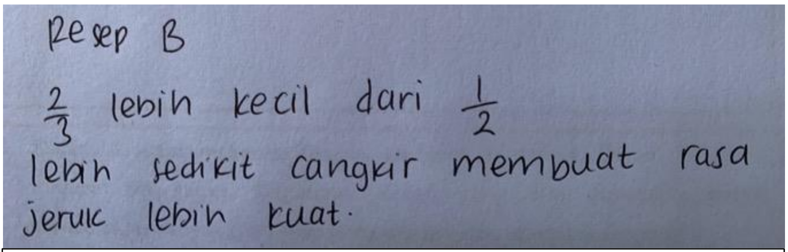


Recipe B



Learning Obstacle

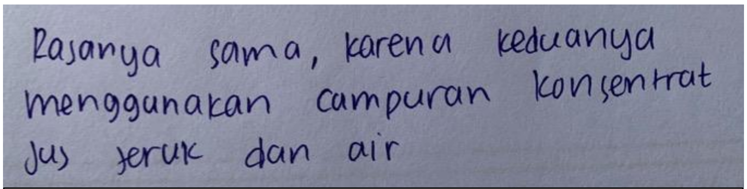
The researcher identifies the learning obstacles students face in relation to Problem 7. The first learning obstacle involves a misunderstanding of the comparison concept. The student's answer is Recipe B, based on the incorrect reasoning that 2/3 is less than 1/2, or that a smaller total number of cups results in a stronger orange flavor. As shown in Figure 25.



Recipe B
$\frac{2}{3}$ is less than $\frac{1}{2}$
Fewer cups make the orange flavor stronger

Figure 25. Learning obstacle 1 in problem 7

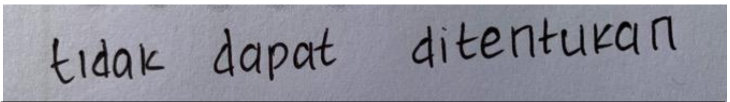
The second learning obstacle involves illogical reasoning on the part of the student. The student concludes that the flavors are the same because both mixtures contain some orange juice concentrate and some water (as shown in Figure 26). The student fails to engage in reasoning based on a pattern or a logical comparison framework.



The flavor is the same, as both use a mixture of orange juice concentrate and water.

Figure 26. Learning obstacle 2 in problem 7

The third learning obstacle involves the failure to employ analytical thinking. The total number of cups used in each mixture differs; thus, the ratios cannot be directly compared by the student. The student answers that it cannot be determined (as shown in Figure 27). The student assumes that specific numerical values must be provided to solve this comparison problem. The exploration of information here could involve understanding relationships or transformations of quantities without requiring specific numerical values, as the diagram provided already represents the quantities. However, the student fails to analyze the comparative information in the diagram to solve the problem.

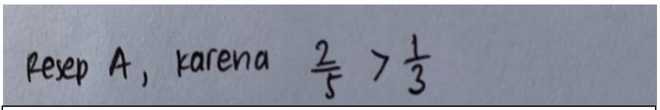


Cannot be determined.

Figure 27. Learning obstacle 3 in problem 7

Technique

The correct technique used by the student, as shown in Figure 28, involves answering Recipe A because 2/3 > 1/2. Alternatively, 2/5 > 1/3, or a greater total number of cups results in a stronger orange flavor.



Recipe A, because $\frac{2}{3} > \frac{1}{3}$

Figure 28. Answer to problem 7

Problem 8

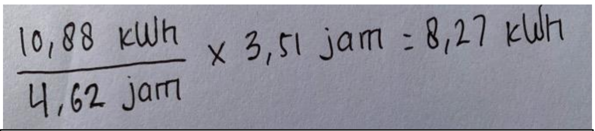
The task type in Problem 8 is a missing value problem. The scientific context involves the energy used by an air conditioner (AC) to operate, where the unit rate is not given, and no integer factor change is present between the provided ratio pairs.

Task

There is a proportional relationship between the operating time of the AC and the amount of energy it uses. When the AC operates for 3.51 hours, it uses 10.88 kWh of energy. How much energy does the AC use when it operates for 4.62 hours?

Learning obstacle

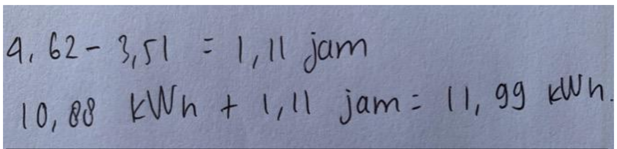
The researcher identifies the learning obstacles students face in relation to Problem 8. The first learning obstacle involves an incorrect approach to the unit rate. As shown in Figure 29, the student answers (kWh of energy) = (10.88 kWh / 4.62 hours) × (3.51 hours) = 8.27 kWh.



$$\frac{10.88\ kWh}{4.62\ hours} > 3.51\ hours = 8.27\ kWh$$

Figure 29. Learning obstacle 1 in problem 8

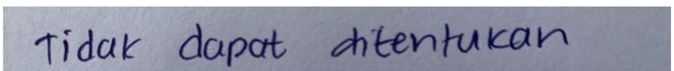
The second learning obstacle involves an incorrect additive strategy. The difference in hours, 4.62 – 3.51 = 1.11 hours, is added to the kWh of energy used when the AC operates for 3.51 hours. Therefore, 10.88 + 1.11 = 11.99 (as shown in Figure 30).



4.62 – 3.51 = 1.11 *hours*
10.88 *kWh* + 1.11 *hours* = 11.99 *kWh*

Figure 30. Learning obstacle 2 in problem 8

The third learning obstacle involves the student's failure to reason quantitatively. The student answers that it cannot be determined (as shown in Figure 31). The presence of quantitative elements (decimal numbers) in this task makes it a problem that the student is unable to solve.

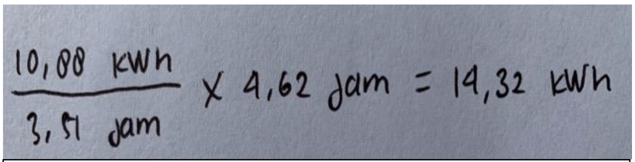


Cannot be determined.

Figure 31. Learning obstacle 3 in problem 8

Technique

The correct technique is demonstrated in the student's answer shown in Figure 32: (kWh of energy) = (10.88 kWh / 3.51 hours) × (4.62 hours) = 14.32 kWh.



$$\frac{10.88\ kWh}{3.51\ hours} \text{x}\ 4.62\ hours = 14.32\ kWh$$

Figure 32. Answer to problem 8

Based on their responses to the learning obstacle tool, seventh-grade students encountered a number of difficulties understanding the concept of proportionality. According to Brousseau (1997) and Sensevy et al. (2008), these learning barriers were further separated into three theoretical categories—epistemological, ontogenic, and didactical obstacles—based on Chevallard's praxeological model.

1. Epistemological Obstacles

The nature of mathematical knowledge itself gives rise to epistemological obstacles. Students' incorrect use of additive reasoning in situations requiring multiplicative thinking was demonstrated by a number of problems. Students' attempt to solve a proportionality task in Problem 1 by estimating the number of calories in 20 grams using additive logic, such as "20 + 40 = 60," demonstrates a profound ignorance of proportional invariance. Students also misapplied proportional reasoning to a non-proportional context in Problem 3, indicating that they lacked the conceptual tools necessary to discern between proportional and non-proportional relationships. Numerous students misunderstood the meaning of unit rates in Problem 5, demonstrating a lack of comprehension regarding the relationships between proportional quantities.

2. Ontogenic Obstacles

Ontogenic obstacles are associated with the developmental stages and past knowledge of individual learners. Students' incapacity to derive or apply general rules involving unit rates was demonstrated in Problem 2. Some used additive language to describe relationships, such as "grams + 40 = calories," indicating a conceptually immature grasp of multiplicative structures. Students frequently confused part-part and part-whole relationships when comparing compound ratios in Problem 7. As is common in the early phases of algebraic thinking, these mistakes imply that the students had not yet acquired the abstract reasoning abilities required to flexibly interpret and manipulate proportional relationships.

3. Didactical Obstacles

The way that mathematical material is taught and presented is the source of didactic obstacles. Students' inability to engage with qualitative comparisons, such as comparing relative quantities in the absence of explicit numbers, was demonstrated in Problem 4, which made them believe the problem was intractable. Students used improper scaling techniques in Problem 6, presumably as a result of their lack of experience with scaling in relevant contexts. Assuming that doubling one quantity should always double the other, even in cases where the relationship was more complex, was a common error.

Lastly, Problem 8 demonstrated how hard it is for students to reason in decimal contexts. A lack of exposure to proportion tasks involving non-whole numbers may be the reason why many people avoided decimal calculations or provided inaccurate estimations.

Didactic design for the concept of proportionality

This section provides an explanation of the entire sequence of tasks presented in the didactic design of the concept of proportionality. In general, this didactic design is presented using praxeology, which includes task designs and the various techniques employed to solve these tasks (praxis), as well as justifications for the techniques in the form of technology and theory (logos). Praxeology is employed here with the aim that the sequence of tasks presented will encourage students to utilize various techniques that assist in acquiring knowledge aligned with scholarly knowledge, thus helping form students' early algebraic thinking.

Task design 1

The first task design is intended for students to form the proportionality constant. Prior to this, students have learned about ratios. In this context, students use their prior knowledge of ratios to form unit ratios. This unit ratio is an invariant ratio that defines the multiplicative relationship between the values of y and x. The value of m, represented by y/x, is the proportionality constant. Conversely, 1/m is also a proportionality constant, expressed as x/y, which can be used to represent the proportional relationship between x and y. Therefore, there are two invariant ratios that determine a specific proportional situation (see Table 1). The contextual problem provided is familiar to the students: the concept of speed, which is closely related to their daily lives and represents two quantities (distance and time) that are also studied in mathematics. Subsequently, the ratio in this context is referred to as speed. Each unit speed ratio has a distinct interpretation.

Table 1. Praxiology of task design 1

Task	Technique	Technology	Theory
Joni and Zio are running around a track at the same speed. Joni takes 20 minutes to run a distance of 4 km. Determine the two-unit speeds that describe Joni's speed!	Compare the distance and time taken: the speed describing Joni's pace is 20 minutes per 4 km and 4 km per 20 minutes. Then, convert each speed to a unit speed: the unit speed is 5 minutes per 1 km and 0.2 km per 1 minute.	Unit speed is interpreted as the quantity of y per 1 x, or the quantity of x per 1 y.	The proportionality constant, m, is an invariant speed that defines the multiplicative relationship between the values of y and x. The unit ratio is the quotient linking the units of measurement for y and x. The value of m is a unit quantity expressed as the simple division $m = y/x$ (Thompson, 1994). The inverse of the proportionality constant, $1/m$, also represents a proportionality constant, x/y, which can be used to describe the proportional relationship between x and y. Therefore, the two invariant ratios (y/x or x/y) determine a specific proportional situation, with each ratio carrying a distinct interpretation (Vergnaud, 2009).

Task design 2

After establishing the unit rate in the first design, the same context is presented with a problem that involves a proportional situation that needs to be solved. This problem is designed to be approached by students using a variety of strategies. The second task can be completed using eight techniques, including: rate tables, unit rate, factor change, function approach, fraction strategy, equivalent fractions, four proportions, and connection of standard algorithms with unit rate and factor change approaches. The intention is to foster students' early algebraic thinking through the concepts of proportions and proportional reasoning, which encompass higher algebra concepts such as invariance and covariance, equivalence classes, and linear functions.

The researcher consistently presents speed-related problems to ensure students' thinking is not interrupted by a change in context until their conception is formed. The use of simple integers is intended not to complicate the calculations, as the focus here is not on computation using standard algorithms but on early algebraic thinking with meaning before computation takes place. Specifically, students are expected to connect standard algorithms with unit rates and factor change approaches through proportional reasoning.

Task 2

Joni and Zio run around a track equally fast. It takes Joni 20 minutes to run 4 km. How long does it take Zio to run 12 km? Work with different strategies!

After students have practiced using the rate table (Technique 1, see Table 2) to understand how time and distance relate in a proportional situation, the next step is to explore unit rates (Technique 2, see Table 3). In this phase, students will focus on simplifying the rate to a per unit value, which offers a more direct connection to the problem at hand. The unit rate provides a more granular approach, allowing students to understand the relationship in terms of a single unit, thus enhancing their ability to apply the concept to more complex scenarios, like Zio's 12 km run.

Table 2. Praxiology of task design 2 (Technique 1)

<table>
<tr><th>Technique 1</th><th>Technology 1</th><th>Theory 1</th></tr>
<tr><td>By using the rate of 20 minutes per 4 kilometers, students can extend this to the rate of 60 minutes per 12 kilometers through addition, and then further expand it using multiplication.
<table>
<tr><td>Km</td><td>4</td><td>8</td><td>12</td></tr>
<tr><td>Menit</td><td>20</td><td>40</td><td>60</td></tr>
</table>
<table>
<tr><td>Km</td><td>4</td><td>12</td></tr>
<tr><td>Menit</td><td>20</td><td>60</td></tr>
</table>
</td><td>This demonstrates multiplication both within and across measurement spaces, which can be useful when students construct proportions that describe the previous situation in terms of the standard algorithm description.</td><td>In a proportional situation, there is a scalar multiplication relationship within the measurement space, and an invariant multiplication relationship between measurement spaces (Schwartz, 1988; Kaput & West, 1994).</td></tr>
</table>

Table 3. Praxiology of task design 2 (Technique 2)

Technique 2	Technology 2	Theory 2
Unit Rate If the rate is 20 minutes per 4 kilometers, then the unit rate is 5 minutes per 1 kilometer. If Zio runs 12 kilometers, the time required will be (12 km × 5 minutes/km) = 60 minutes.	This strategy involves two steps: division to determine the unit rate and then multiplication to determine the missing value. When using the unit rate, the first step is to identify which unit rate corresponds to the question.	In a proportional situation, two invariant unit rates/proportionality constants exist across the measurement space. Quantitatively, the proportionality constant defines an equivalence class that corresponds to all pairs of rates (x, y) in a given proportional situation (Lobato & Siebert, 2002). The invariant relationship between two variables, x and y, can be extended to other equal multiples of x and y using proportional reasoning (Lamon, 2007).

Building upon the unit rate approach, we now move to the concept of a "factor of change" (Technique 3, see Table 4). In this technique, students will use the multiplier derived from the unit rate to scale the solution proportionally. By multiplying the time needed for a smaller distance (4 km) by the factor, students can determine the time needed for a greater distance (12 km). This method emphasizes the scaling of quantities and deepens the students' understanding of proportional reasoning as it applies to real-world scenarios.

Table 4. Praxiology of task design 2 (Technique 3)

Technique 3	Technology 3	Theory 3
Factor of Change If it takes 20 minutes to run 4 kilometers, then to run 3 times the distance (12 kilometers), Zio will need to run 3 times the amount of time required previously: 20 minutes × 3 = 60 minutes.	A scalar factor does not have associated units. Multiplying, tripling, and similar operations are scalar operations. This strategy relies on the multiplication relationship within the measurement space. The scalar factor is not constant, and a constant factor is not required (meaning we can determine the time of 40 minutes to run 8 kilometers by multiplying, rather than by tripling).	In a proportional situation, there is a scalar multiplication relationship within the measurement space. With multiplicative reasoning and alternative quantitative representations, this describes how two quantities are related in covariance, changing together while maintaining equality between pairs of rates in a proportional situation (Thompson & Thompson, 1994; Oehrtman et al., 2008).

Once students are comfortable with the concept of a factor of change, the next logical progression is the use of functions to model the relationship between time and distance (Technique 4, see Table 5). Here, students will transition from discrete multiplicative reasoning to continuous functional thinking. By representing the problem with a function, students will not only calculate time for various distances but also understand the underlying algebraic structure that governs proportional relationships. This approach introduces the idea of a linear function, where the rate of change is constant and directly tied to the unit rate established earlier.

Table 5. Praxiology of task design 2 (Technique 4)

Technique 4	Technology 4	Theory 4
Function Approach Using the information provided in the problem, the explicit function is written in relation to time and distance through the equation t = (5 minutes per km) × s. This relationship is used to calculate the time Zio needs to run a distance of 12 km.	This is a constant function rule between units of measurement (from kilometers to minutes). The unit rate plays an important role in such a rule, resembling a constant proportionality.	Proportionality is a linear relationship between two quantities that covary according to the model $y=mx$, where m is the unit rate. All corresponding rate pairs (x,y) lie on the graph of the line $y=mx$, which passes through the origin (Lamon, 2007). The proportionality constant (m) is graphically represented as the slope of the line for the graph of $y=mx$ (Lobato & Siebert, 2002).

Having established a function to model the relationship between time and distance, we now turn to fractions and equivalent fractions (Technique 5, see Table 6). This technique shifts the focus from functional equations to fractional reasoning. By recognizing that a proportion can be expressed as a fraction, students develop a deeper understanding of how proportional relationships can be manipulated algebraically. The use of equivalent fractions aligns with the concept of scaling and offers another perspective on solving proportional problems.

Table 6. Praxiology of task design 2 (Technique 5)

Technique 5	Technology 5	Theory 5
The strategy involves creating equivalent fractions without explicitly stating the units and applying the method of multiplying both the numerator and denominator by the same factor, in this case, 3, to find the equivalent fraction. $\frac{20}{4}=\frac{x}{12}$ $x=60$	With this approach, unit measurement space is not used, but reasoning with rational numbers is applied. This approach removes the content, and students are required to make the connection back to the problem themselves.	The first element in a fraction is the numerator, and the second element is the denominator. The pair representing the fraction is typically written in the fraction bar notation m/n. The numerator and denominator are multiplied by the same factor to obtain an equivalent fraction (Behr et al., 1992).

From the manipulation of fractions, we now move to the more formal use of equivalent fractions in proportion problems (Technique 6, see Table 7). This technique focuses on setting up proportions, solving for missing values using cross-multiplication, and interpreting the results. The process reinforces the idea of equivalence between ratios and extends the students' understanding of the earlier methods to a broader set of proportional relationships. By emphasizing proportional reasoning through cross-multiplication, students solidify their grasp of the connections between ratios and real-world applications.

Table 7. Praxiology of task design 2 (Technique 6)

Technique 6	Technology 6	Theory 6
Equivalent Fractions To form a proportion, determine the product of the multiplication, and solve the equation for the missing value. $\frac{20\ minutes}{4\ km} = \frac{x\ minutes}{12\ km}$ $20\ minutes \,.\, 12\ km = x\ minutes \,.\, 4\ km$ $\frac{20\ minutes \,.\, 12\ km}{4\ km} = x\ minutes$ $60\ minutes = x\ minutes$	Units are not eliminated when a proportion is formed, but they are ignored when performing cross-multiplication. The equivalence of rate pairs allows us to form proportions because all rate pairs are equivalent. This approach is restructured and connected to the concept of unit rates and the factor-change approach.	A fraction is an equivalence class of ordered pairs of integers that satisfy axioms 1-4 (Freudenthal, 1983). **Axiom 2 – Fractions** Two ordered pairs (*m*, *n*) and (*p*,*q*) are equivalent if and only if *mq*=*np*. Axiom 2 formalizes this reasoning by using the well-known cross-multiplication rule, which implicitly defines equivalent pairs. In fraction bar notation, the pairs *m/n* and *p/q* are equivalent if and only if *mq* = *np*. All sets of equivalent pairs (an equivalence class) are considered as a single fraction. Two fractions are considered equal if their equivalence classes are the same. A particular pair located within an equivalence class is called the *representative* of the fraction (Vergnaud, 1983).

Once students are familiar with solving proportions using equivalent fractions, the next step is to explore the concept of multiple proportional relationships through the use of several distinct proportions (Technique 7, see Table 8). This technique allows students to see how different ratios can describe the same relationship, providing a richer understanding of proportional reasoning. By working with multiple proportions, students can apply their knowledge of unit rates and equivalent fractions in a more flexible manner, understanding how different relationships interconnect.

Table 8. Praxiology of task design 2 (Technique 7)

Technique 7	Technology 7	Theory 7
4 Proportions Here are 4 different proportions showing the relationship between the running speeds of Joni and Zio. $\frac{20\ minutes}{4\ km} = \frac{60\ minutes}{12\ km}$ $\frac{4\ km}{20\ minutes} = \frac{12\ km}{60\ minutes}$ $\frac{20\ minutes}{60\ minutes} = \frac{4\ km}{12\ km}$ $\frac{60\ minutes}{20\ minutes} = \frac{12\ km}{4\ km}$	To connect each proportion with the multiplication relationship within and between units of measurement (km and minutes), we use the multiplication rule to demonstrate how something changes (covariance) and something remains constant (invariance).	Identifying and utilizing the relationships of covariance and invariance, along with multiplicative thinking, are the core aspects of proportional reasoning (Lamon, 2007; van den Heuvel-Panhuizen, 2003).

Finally, after exploring multiple proportions, students are introduced to the concept of connecting standard algorithms with unit rates and scaling factors (Technique 8, see Table 9). This final technique consolidates the earlier methods, allowing students to synthesize their understanding and apply it in various problem-solving contexts. By linking algorithms with the principles of unit rates and factors of change, students are equipped to handle more complex problems and to understand the underlying algebraic and proportional principles that govern the solution processes.

Table 9. Praxiology of task design 2 (Technique 8)

Technique 8	**Technology 8**	**Theory 8**
4 Proportions Here are 4 different proportions showing the relationship between the running speeds of Joni and Zio. $\frac{20\ minutes}{4\ km} = \frac{60\ minutes}{12\ km}$ $\frac{4\ km}{20\ minutes} = \frac{12\ km}{60\ minutes}$ $\frac{20\ minutes}{60\ minutes} = \frac{4\ km}{12\ km}$ $\frac{60\ minutes}{20\ minutes} = \frac{12\ km}{4\ km}$	Although cross-multiplying kilometers (km) by minutes cannot be directly interpreted as a meaningful operation, we can guide the interpretation back to the problem by identifying the unit rate and the factor of change.	In unit rate proportions, we observe an invariant relationship between measurement spaces, or across time and distance within rate pairs. In scalar proportions, we observe a multiplicative relationship within the measurement space of rate pairs. (Kaput & West, 1994; Thompson, 1994).

Task design 3

Proportional reasoning with a unit rate and various strategies to solve problems through algebraic thinking was developed in Task Design 2. Task Design 3 is intended for students to solve problems using the appropriate proportionality constant (see Table 10). Continuing with the same context, students are prompted to think about how the approach to the unit rate changes. There are two-unit rates/proportionality constants: $1/m$, which is also a proportionality constant, and x/y, which can be used to express the proportional relationship between x and y. Therefore, in a given proportional situation, the unit rate can be selected according to the appropriate interpretation.

Table 10. Praxiology of task design 3

Task	**Technique**	**Technology**	**Theory**
How many kilometers will Zio cover in 60 minutes? Write down the rule that can be used to determine the distance traveled.	Connection with Unit Rate $\frac{20\ minutes}{4\ km} = \frac{60\ minutes}{x\ km}$ $4\ km \,.\, 60\ minutes = x\ km \,.\, 20\ minutes$ $\frac{4\ km \,.60\ minutes}{20\ minutes} = x\ km$ $60\ minutes \,.\frac{4\ km}{20\ minutes} = x\ km$ $60\ minutes \,.\frac{0{,}2\ km}{1\ minute} = x\ km$ $12\ km = x\ km$	The standard algorithm strategy is connected with the unit rate. When using the unit rate, the first step is to determine which unit rate is appropriate for the question. The unit rate plays an important role in the rule of the constant function between measurement spaces.	The reciprocal of the proportionality constant, $1/m$, is also a proportionality constant, x/y, which can be used to express the proportional relationship between x and y (Vergnaud, 2009). Therefore, two invariant rates (y/x or x/y) define a specific proportional situation, with each rate having a distinct interpretation. In the proportion of unit rates, we observe the invariant relationship between measurement spaces, or across time and distance

Task	Technique	Technology	Theory
			pairs of rates (Thompson, 1994).
	Connection with the Factor of Change $$\frac{20\ minutes}{4\ km} = \frac{60\ minutes}{x\ km}$$ $$4\ km\ .\ 60\ minutes\ =\ x\ km\ .\ 20\ minutes$$ $$\frac{4\ km\ .\ 60\ minutes}{20\ minutes} = x\ km$$ $$4\ km\ .\ \frac{60\ minutes}{20\ minutes} = x\ km$$ $$4\ km\ .\ 3 = x\ km$$ $$12\ km = x\ km$$	The standard algorithm strategy is connected with the factor of change.	In scalar proportions, we observe the multiplication relationship within the measurement space of rate pairs (Kaput & West, 1994).
	Function Approach $y = m\ .\ x$ s (km) = 0,2 km/min . *t* (min)	The unit rate plays an important role in the rule of the constant function between measurement spaces.	Proportionality is a linear relationship between two covarying quantities according to the model $y = mx$, where m is the unit rate (Lamon, 2007).

Design of task 4

The design of Task 4 (see Table 11) is intended to provide a qualitative distraction to students by presenting a non-proportional situation. With the experience gained from completing Task 3 and the proportional reasoning developed, students can broaden their understanding and procedures for both proportional and non-proportional situations. Additionally, students are not expected to compute immediately before understanding the problem and thinking about how to solve it. In the analysis of learning obstacles, many secondary school students perform cross-multiplication in non-proportional situations. To minimize this learning obstacle, Task 4 should be presented to elementary school students when learning the concept of proportion.

The problem context is familiar—running around a track—similar to the speed context discussed earlier. The use of simple numbers ensures that the students' focus remains on proportional reasoning rather than being hindered by complicated computations.

Table 11. Praxiology of Task Design 4

Task	Technique	Technology	Theory
Joni and Zio are running at the same speed around a track. Joni starts running first. When Joni has completed 9 laps, Zio has completed only 3 laps. When Zio completes 15 laps, how many laps has Joni completed?	Joni has run 6 more laps than Zio, so: Joni Zio 9 +6 3 ? +6 15 15 + 6 = 21	If the situation is proportional, there is a unit rate and an integer factor of change. This is a quantitative distraction that can lead to incorrect proportional reasoning being applied to a non-proportional task.	In proportional situations, there is a scalar multiplication relationship within the measurement space, and an invariant multiplication relationship between the measurement spaces (Schwartz, 1988). Multiplicative thinking is at the core of proportional reasoning processes (Vergnaud, 1983).

Reflection

Traditionally, proportional situations have been represented in the mathematics curriculum through the use of proportions, where two pairs of rates are related by equality, A/B = C/D.

Research has shown that students who can solve problems involving proportions do not necessarily reason proportionally. The traditional curriculum's treatment of proportionality, which defines proportions as equivalent pairs of rates and uses cross-multiplication and division to solve for missing values, is often poorly understood and disconnected from students' informal understanding and natural operations in proportion-related situations.

A different approach to teaching proportionality and proportional reasoning must be developed and implemented to better support the development of these understandings and processes. This approach should encompass a variety of proportion-related situations and tasks, while providing sufficient time and experience for students to build their understanding and reasoning processes related to proportions through intuitive strategies before more procedural approaches are introduced. The design of Tasks 2 and 3 explores a new approach to constructing understanding of proportions and proportional reasoning processes in the mathematics classroom. This can help avoid one of the obstacles that arise, such as in Task 4, where students may immediately apply cross-multiplication without an understanding of proportionality.

Task Design 5

Task Design 5 is intended to expand students' understanding of proportionality, thus enhancing their algebraic reasoning (see Table 12). It is essential to present problems that challenge students' existing concepts, guiding them toward a more scientific conception. Students' conceptions can only change when they encounter situations they cannot resolve with their current understanding. This task design enables instructors to assist students in accommodating new perspectives and procedures, specifically those involving rational numbers. Task Design 5 provides an opportunity for students to analyze concepts more deeply, revising or extending their existing conceptions.

In completing Tasks 2 and 3, students will encounter strategies for equivalent fractions. Task Design 5 introduces an opportunity to revise and broaden students' understanding of proportionality, emphasizing that proportion is a statement about two equivalent ratio representations, not merely limited to equivalent fractions.

The researcher introduces a new problem context involving the energy used by an air conditioner (AC). This context serves to broaden students' perspectives, as they have already developed proportional reasoning. The quantities involved—energy and time—are scientific concepts students are familiar with. Simple integers are used in the problem, leading to the emergence of rational numbers.

Table 12. Praxiology of Task Design 5

Task	Technique	Technology	Theory
There is a proportional relationship between the time an air conditioner (AC) operates and the energy it consumes. When the AC operates for 2 hours, it uses 7 kWh of energy. How much energy will the AC use when it	Equivalent Ratios $\frac{2}{7}=\frac{3}{x}$ $x=\frac{7.3}{2}$ $x=10{,}5$ $\frac{2}{7}=\frac{3}{10{,}5}$	The rational equation $\frac{2}{7}=\frac{3}{x}$ and the ratio comparison $2:7 \sim 3:x$ has a solution for x, which is calculated as $x=\frac{7.3}{2}=10{,}5$. However, when expressed as a proportion in the form of a fraction, the equation does not have a solution, as the pair (3; 10,5) does not represent a valid fraction. On the other hand, $\frac{3}{10{,}5}$ is not a fraction based on its	**Definition 1R.** A ratio is an equivalence class of sequences of real numbers that satisfy axioms 1R and 2R. (Hartnett, 2012) **Axiom 1R.** There are no zero elements in the sequence. **Axiom 2R.** (Cross Multiplication Rule) Two sequences $(a_1, a_2, ..., a_n)$ and $((b_1, b_2, ..., b_n)$ are equivalent: $(a_1, a_2, ..., a_n) \sim (b_1, b_2, ..., b_n)$ if $a_ib_j = a_jb_i$, for all $1 \leq i,j \leq n$. The ratio of two elements, such as $a_1 : a_2$, is crucial in applications. Each such ratio has a unique representative in its lowest terms, $b_1 : 1$. Since the two representatives are equivalent, $a_1 : a_2 \sim b_1 : 1$, it allows for the consideration of the entire sequence of equivalent ratios of two elements $a_1 : a_2$ as the real number b_1.

Task	Technique	Technology	Theory
operates for 3 hours?		definition: a fraction is an equivalence class of ordered pairs of integers that satisfy axioms 1-4.	In some applications of the ratio of two elements, the multiplication of ratios can consistently be defined as $(a_1 : a_2)\ (c_1 : c_2) = a_1c_1 : a_2\ c_2$, similar to the multiplication of fractions. The result is the same as the product of the two real numbers that represent the ratios and is independent of the choice of representatives (Hartnett, 2012).

Reflection

Proportion is the statement that two ratio representations are equivalent. For example, 1:2 ~ 4:*x*, where the symbol "~" signifies equivalence. (In this context, the definition of writing a proportion as a fraction, such as 1/2 ~ 4/*x* or 1/2 = 4/*x*, is incorrect.) Axiom 2R justifies the solution method. For instance, two ordered pairs (1, 2) and (4, *x*) are equivalent if $1 \cdot x = 2 \cdot 4$, hence x = 8. The notation and reasoning associated with this definition of proportion are typically not employed in school algebra. Instead, it is common practice to write 1/2 = 4/*x* and treat it as an equation between two fractions or ratios. It can also be observed that some textbooks define proportion in terms of fractions. In such cases, all terms of the proportion, by definition, must be integers. Otherwise, rational equations come into play, displacing proportions.

It is emphasized that while there may be some similarities, ratios are not fractions. They are distinct mathematical objects arising from different real-world problems, consisting of different elements (real numbers for ratios versus integers for fractions) and satisfying different axioms. Ratios (except for the ratio of two elements) cannot be ordered, added, or multiplied.

Five task designs that adhered to praxeological principles were developed using the identified obstacles as a basis. Contextual variation, cognitive scaffolding, and opportunities for mathematical reflection were all incorporated into each task to address particular learning challenges. By assisting students in developing more resilient, adaptable, and theoretically sound proportional reasoning skills, these task designs offer a strategic response to the observed challenges.

Discussion

In order to enhance students' conceptual grasp of proportionality, this study sought to develop a didactic design grounded in mathematical praxeology and the Anthropological Theory of the Didactic (ATD). The study found a number of learning challenges that students frequently face when acquiring this concept through qualitative analysis of student responses.

The learning obstacles identified in this study can be divided into ontogenic, didactical, and epistemological obstacles using Brousseau's (1997) framework.The nature of mathematical knowledge itself is the source of epistemological obstacles. Students in this study showed a conceptual misunderstanding of proportional relationships by regularly using additive reasoning in situations that called for multiplicative thinking. This supports research by Misailidou and Williams (2003), who demonstrated that students frequently turn to additive strategies as a result of a lack of understanding of proportional structures.

Didactical obstacles have to do with teaching methods and how mathematical ideas are explained. The students struggled with less familiar representations like tables or ratio strips and were perplexed by missing or implicit information in a number of problem scenarios. These pedagogical concerns align with Lobato et al. (2010), who contend that a lack of exposure to diverse contexts and representations results in a brittle comprehension of proportionality.

Conversely, ontogenic obstacles result from the unique attributes of each learner, including affective factors, developmental readiness, and past knowledge. Because of their poor sense of number or preconceived notions from previous education, some of the students in this study had trouble using proportional reasoning. These results are consistent with Radford (2014), who highlights how crucial it is to consider the learner's developmental trajectory when it comes to mathematical acquisition. Furthermore, students' willingness to participate in cognitively demanding proportional tasks was found to be hampered by their affective responses, such as anxiety related to mathematics or a lack of confidence in their ability to solve problems.

Based on these obstacles, the study suggested a didactic design that included tasks, techniques, technologies, and theory and was organized according to the praxeological framework (Chevallard, 1989). Five structured learning tasks are included in the design, which was created to assist students in developing early algebraic thinking by removing instructional and conceptual obstacles pertaining to proportional reasoning.

Despite not being used in this study, the didactic design was developed using empirical data from student interviews and diagnostic tasks. Every task was designed to directly address a particular ontogenic, didactical, or epistemological barrier. To encourage proportional interpretation and disrupt additive reasoning, for instance, a task that required students to compare quantities in real-world situations was created. In order to help students generalize multiplicative structures across contexts, another task included multiple representations of ratios. Tasks were scaffolded to allow for gradual cognitive engagement to address ontogenic obstacles. They also included elements that foster student motivation and confidence, such as relatable contexts and opportunities for peer collaboration.

This design strategy supports Artigue's (2009) assertion that developing effective instructional interventions requires an understanding of learning barriers. Additionally, it supports findings from recent research (e.g., Van Dooren et al., 2010; Boyer & Levine, 2015), which emphasize the value of scaffolding proportional reasoning through contextualized and cognitively challenging problems.

The study's conclusions have several significant ramifications for how mathematics education is planned, especially in Indonesian classrooms. First, teachers and curriculum designers can more accurately diagnose students' challenges and respond with focused, research-based instructional strategies by utilizing a theoretical framework to identify and classify learning obstacles.

Second, this study's praxeology-based design helps students transition from procedural to conceptual understanding by emphasizing not only how to solve proportional problems but also why specific approaches are effective. This is consistent with Chevallard's (2006) focus on teaching mathematics by combining praxis (technique) and logos (justification).

Third, the task design's integration of relevant, real-world contexts enables students to make connections between newly learned material and past experiences, which promotes the growth of early algebraic thinking. By acknowledging the various cognitive and emotional beginning points of students, addressing ontogenic barriers within this framework also promotes inclusive teaching.

Even though this study produced a didactic design that is theoretically supported and empirically informed, it is crucial to remember that the design was neither used nor assessed in a classroom. The goal of future research should be to test the suggested design in actual classrooms to assess its efficacy and make necessary adjustments in light of implementation results. Furthermore, more research could examine the ways in which other elements—like classroom discourse, student motivation, or sociocultural background—influence the development of proportional reasoning.

Conclusion

To improve students' comprehension of proportionality in lower secondary education, this study effectively created a didactic design influenced by mathematical praxeology and the Anthropological Theory of the Didactic (ATD). The design was developed based on a thorough examination of the learning challenges that students faced, which were divided into three categories using Brousseau's (1997) framework: ontogenic obstacles, which refer to challenges originating from a learner's prior knowledge, developmental readiness, or affective factors; didactic obstacles, which include students' confusion when performing tasks that are unfamiliar or implicitly structured; and epistemological obstacles, which include the misuse of additive strategies in proportionate contexts.

The study’s primary outcome is the identification and analysis of these three categories of barriers, which form the basis of an instructional design with a theoretical underpinning. The didactic design was developed using empirical data, such as diagnostic tasks, classroom observations, and student-teacher interviews, even though it hasn't been used yet. In order to promote early algebraic thinking and proportional reasoning, each of the five constructed task designs was created to target epistemological, didactical and ontogenic obstacles.

The study also highlights how crucial it is to create contextualized learning opportunities that relate mathematical concepts to students' real-world situations. This kind of contextualization promotes deeper engagement with the underlying mathematical structures in addition to helping students connect new ideas to what they already know. This supports Chevallard's (1989) claim that praxis and logos, the theoretical and practical aspects of knowledge, must be integrated in mathematics instruction.

The study has certain limitations in spite of these contributions. The effectiveness of the suggested didactic design has yet to be confirmed by further implementation, primarily because it was not tested in real classrooms. Additionally, the study didn't look into how outside variables like classroom dynamics, parental participation, and student motivation might impact how well instructional interventions work. Given their strong ties to learners' individual experiences and socioemotional environments, these factors are particularly pertinent when addressing ontogenic barriers.

Future studies should concentrate on putting the suggested design into practice, assessing it in various learning environments, and determining whether it can be applied to mathematical domains other than proportionality. Longitudinal studies could also evaluate how ontogenic factors change over time in various learning environments and how such designs affect students' mathematical thinking.

This study concludes by emphasizing the importance of identifying ontogenic, didactical, and epistemological learning barriers as a precondition for effective instructional design. Teachers can create more effective teaching strategies that not only address students' misconceptions but also lay the groundwork for advanced mathematical understanding by referencing strong theoretical frameworks and basing design choices on qualitative data.

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